

GEODESICS IN MARGULIS SPACETIMES

WILLIAM M. GOLDMAN AND FRANÇOIS LABOURIE

Dedicated to the memory of Dan Rudolph

ABSTRACT. Let M^3 be a Margulis spacetime whose associated complete hyperbolic surface Σ^2 has compact convex core. Generalizing the correspondence between closed geodesics on M^3 and closed geodesics on Σ^2 , we establish an orbit equivalence between recurrent spacelike geodesics on M^3 and recurrent geodesics on Σ^2 . In contrast, no timelike geodesic recurs in either forward or backwards time.

CONTENTS

Introduction	1
1. Geodesics on affine manifolds	3
2. Flat Lorentz 3-manifolds	3
3. From geodesics in Σ^2 to geodesics in M^3	6
4. Appendix: Cohomology and positive functions	9
References	11

INTRODUCTION

A *Margulis spacetime* is a complete flat affine 3-manifold M^3 with free nonabelian fundamental group Γ . It necessarily carries a unique parallel Lorentz metric. Parallelism classes of timelike geodesics form a noncompact complete hyperbolic surface Σ^2 . This complete hyperbolic surface is naturally associated to the flat 3-manifold M^3 and we regard M^3 as an *affine deformation* of Σ^2 . This note relates the dynamics of the geodesic flow of the flat affine manifold M^3 to the dynamics of the geodesic flow on the hyperbolic surface Σ^2 .

We restrict to the case that Σ^2 has a compact convex core (that is, Σ^2 has finite type and no cusps). Equivalently, the Fuchsian group Γ_0 corresponding to $\pi_1(\Sigma^2)$ is *convex cocompact*. In particular Γ_0 is finitely generated and contains no parabolic elements. Under this assumption,

Date: January 11, 2013.

every free homotopy class of an essential closed curve in Σ^2 contains a unique closed geodesic. Since Σ^2 and M^3 are homotopy-equivalent, free homotopy classes of essential closed curves in M correspond to free homotopy classes of essential closed curves in Σ^2 . Every essential closed curve in M^3 is likewise homotopic to a unique closed geodesic in M^3 .

In her thesis [4, 8], Charette studied the next case of dynamical behavior: geodesics spiralling around closed geodesics both in forward and backward time. She proved bispiralling geodesics in M^3 exist, and correspond to bispiralling geodesics in Σ^2 .

This paper extends the above correspondence between geodesics on Σ^2 and M^3 to recurrent geodesics.

A geodesic (either in Σ^2 or in M^3) is *recurrent* if and only if it (together with its velocity vector) is recurrent in *both* directions. These correspond to recurrent points for the corresponding geodesic flows as in Katok-Hasselblatt [17], §3.3. (Our meaning of the term “recurrent” agrees with the term “nonwandering” used by Eberlein [12].) Under our hypotheses on Σ^2 , a geodesic on Σ^2 is recurrent if and only if the corresponding orbit of the geodesic flow is precompact.

Theorem 1. *Let M^3 be a Margulis spacetime whose associated complete hyperbolic surface Σ has compact convex core.*

- *The recurrent part of the geodesic flow for Σ^2 is topologically orbit-equivalent to the recurrent spacelike part of the geodesic flow of M^3 .*
- *The set of recurrent spacelike geodesics in a Margulis spacetime is the closure of the set of periodic geodesics.*
- *No timelike geodesic recurs.*

A semiconjugacy between these flows was observed by D. Fried [13].

This note is the sequel to [15], which characterizes properness of affine deformations by positivity of a marked Lorentzian length spectrum, the *generalized Margulis invariant*. A crucial step in the proof that properness implies positivity is the construction of sections of the associated flat affine bundle, called *neutralized sections*. A further modification of neutralized sections produces an orbit equivalence between recurrent geodesics in Σ and recurrent geodesics in M .

It follows that the set of recurrent spacelike orbits of the geodesic flow is a Smale hyperbolic set in TM .

Null geodesics not parallel to a point in the limit set Λ of Γ_0 do not recur. We do not discuss the recurrence of null geodesics parallel to a point of Λ in this paper, but suspect that such null geodesics do not

recur either. Therefore we conjecture that the only recurrent orbits of the orbit flow are spacelike.

We thank Mike Boyle, Virginie Charette, Suhyoung Choi, Todd Drumm, David Fried, and Gregory Margulis for helpful conversations. We are grateful to Domingo Ruiz for pointing out several corrections.

1. GEODESICS ON AFFINE MANIFOLDS

An *affinely flat manifold* is a smooth manifold with a distinguished atlas of local coordinate systems whose charts map to an affine space \mathbf{E} such that the coordinate changes are restrictions of affine automorphisms of \mathbf{E} . Denote the group of affine automorphisms of \mathbf{E} by $\text{Aff}(\mathbf{E})$. This structure is equivalent to a flat torsionfree affine connection. The affine coordinate atlas globalizes to a *developing map*

$$\tilde{M} \xrightarrow{\text{dev}} \mathbf{E}$$

where $\tilde{M} \rightarrow M$ denotes a universal covering space of M . The coordinate changes globalize to an affine holonomy homomorphism

$$\pi_1(M) \xrightarrow{\rho} \text{Aff}(\mathbf{E})$$

where $\pi_1(M)$ denotes the group of deck transformations of $\tilde{M} \rightarrow M$. The developing map is equivariant respecting ρ .

Denote the vector space of translations $\mathbf{E} \rightarrow \mathbf{E}$ by \mathbf{V} . The action of \mathbf{V} by translations on \mathbf{E} defines a trivialization of the tangent bundle $TM \cong M \times \mathbf{V}$. In these local coordinate charts, a geodesic is a path

$$p \longmapsto p + tv$$

where $p \in \mathbf{E}$ and $v \in \mathbf{V}$ is a vector. In terms of the trivialization the geodesic flow is:

$$\begin{aligned} \mathbf{E} \times \mathbf{V} &\xrightarrow{\tilde{\psi}_t} \mathbf{E} \times \mathbf{V} \\ (p, v) &\longmapsto (p + tv, v) \end{aligned}$$

for $t \in \mathbb{R}$. Clearly this \mathbb{R} -action commutes with $\text{Aff}(\mathbf{E})$.

Geodesic completeness implies that dev is a diffeomorphism. Thus the universal covering \tilde{M} is affinely isomorphic to affine space \mathbf{E} and $M \cong \mathbf{E}/\Gamma$, where $\Gamma := \rho(\pi_1(M))$ is a discrete group of affine transformations acting properly and freely on \mathbf{E} .

2. FLAT LORENTZ 3-MANIFOLDS

Let $\text{Aff}(\mathbf{E}) \xrightarrow{\text{L}} \text{GL}(\mathbf{V})$ denote the homomorphism given by *linear part*, that is, $\text{L}(\gamma) = A$ where

$$p \xrightarrow{\gamma} A(p) + b.$$

The differential of γ at *any* point p identifies with its linear part $L(\gamma)$ via the identification $TM \cong M \times V$.

Any $L(\Gamma)$ -invariant nondegenerate inner product \langle, \rangle on V defines a Γ -invariant flat pseudo-Riemannian structure on E which descends to $M = E/\Gamma$. In particular affine manifolds with $L(\Gamma) \subset O(n-1, 1)$ are precisely the *flat Lorentzian manifolds*, and the underlying affine structures their Levi-Civita connections.

For this reason we henceforth fix the invariant Lorentzian inner product on V , and hence the (parallel) flat Lorentzian structure on E . The group $\text{Isom}(E)$ of Lorentzian isometries is the semidirect product of the group V of translations of E with the orthogonal group $O(n-1, 1)$ of linear isometries. Linear part $\text{Isom}(E) \xrightarrow{L} O(n-1, 1)$ defines the projection homomorphism for the semidirect product. For $l \in \mathbb{R}$, define

$$S_l := \{v \in V \mid \langle v, v \rangle = l\}.$$

When $l > 0$, S_l is a Riemannian submanifold of constant curvature $-l^{-2}$ and when $l < 0$, it is a Lorentzian submanifold of constant curvature l^{-2} . In particular S_{-1} is a disjoint union of two isometrically embedded copies of hyperbolic $n-1$ -space H^{n-1} and S_1 is *de Sitter space*, a model space of Lorentzian curvature $+1$.

The subset $T_l(M)$ consisting of tangent vectors v such that $\langle v, v \rangle = l$ is invariant under the geodesic flow. Indeed, using parallel translation, these bundles trivialize over the universal covering E :

$$T_l(E) \xrightarrow{\cong} E \times S_l$$

Abels-Margulis-Soifer [2, 3] proved that if a discrete group of Lorentz isometries acts properly on Minkowski space E , and $L(\Gamma)$ is Zariski dense in $O(n-1, 1)$, then $n = 3$. Consequently every complete flat Lorentz manifold is a flat Euclidean affine fibration over a complete flat Lorentz 3-manifold. Thus we henceforth restrict to $n = 3$.

Let M^3 be a complete affinely flat 3-manifold. By Fried-Goldman [14], either Γ is solvable or $L \circ h$ embeds Γ as a discrete subgroup in (a conjugate of) the orthogonal group

$$SO(2, 1) \subset GL(3, \mathbb{R}).$$

The cases when Γ is solvable are easily classified (see [14]) and we assume we are in the latter case. In that case, M^3 is a complete flat Lorentz 3-manifold.

In the early 1980's Margulis, answering a question of Milnor [22]), constructed the first examples [19, 20], which are now called *Margulis spacetimes*. Explicit geometric constructions of these manifolds have been given by Drumm [9, 10] and his coauthors [4, 5, 6, 7, 11].

Since the hyperbolic plane \mathbf{H}^2 is the symmetric space of $\mathrm{SO}(2, 1)$, Γ acts properly and discretely on \mathbf{H}^2 . Since M^3 is aspherical, its fundamental group $\pi_1(M^3) \cong \Gamma$ is torsionfree, so Γ acts freely as well. Therefore the quotient $\mathbf{H}^2/\mathrm{L}(\Gamma)$ is a complete hyperbolic surface Σ^2 . Furthermore, by Mess [21], Σ is noncompact. (See Goldman-Margulis [16] and Labourie [18] for alternate proofs.) Furthermore every noncompact complete hyperbolic surface occurs for a Margulis spacetime (Drumm [9])

The points of Σ^2 correspond to parallelism classes of (unoriented) timelike geodesics on M^3 as follows. It suffices to identify \mathbf{H}^2 with the parallelism classes of (unoriented) timelike geodesics in \mathbf{E} , equivariantly respecting $\mathrm{Isom}(\mathbf{E}) \xrightarrow{\sim} \mathrm{SO}(2, 1)$. The velocity of a unit-speed timelike geodesic in \mathbf{E} is a $\tilde{\psi}$ -orbit in

$$\mathrm{T}_{-1}\mathbf{E} \cong (\mathbf{E} \times \mathbf{S}_{-1}).$$

The two components of \mathbf{S}_{-1} correspond to future-pointing timelike geodesics and past-pointing timelike geodesics respectively. Points in \mathbf{S}_{-1} correspond to points in \mathbf{H}^2 (the projectivization of \mathbf{S}_{-1}) together with an orientation of \mathbf{H}^2 . The geodesic flow $\tilde{\psi}$ gives $\mathrm{T}_{-1}\mathbf{E}$ the structure of a principal \mathbb{R} -bundle over the quotient. The quotient identifies with an affine bundle over $\mathbf{S}_{-1} \cong \mathbf{H}^2 \times \{\pm 1\}$ whose associated vector bundle is the tangent bundle, as follows. The space of lines parallel to a fixed timelike vector \mathbf{v} with the quotient affine space, whose underlying vector space is $\mathbf{V}/(\mathbf{v}) \cong (\mathbf{v})^\perp$. The tangent space to \mathbf{S}_{-1} at \mathbf{v} is \mathbf{v}^\perp proving the claim.

Passing to the quotient by Γ :

$$\mathrm{T}_{-1}M \cong (\mathbf{E} \times \mathbf{H}^2)/\Gamma.$$

Since $\Gamma \xrightarrow{\sim} \mathrm{SO}(2, 1)$ is a discrete embedding [14], $\mathrm{SO}(2, 1)$ acting properly on \mathbf{H}^2 implies that Γ acts properly on \mathbf{H}^2 . Cartesian projection $\mathbf{E} \times \mathbf{H}^2 \rightarrow \mathbf{H}^2$ induces a projection

$$\mathrm{T}_{-1}M \longrightarrow \mathbf{H}^2/\mathrm{L}(\Gamma) = \Sigma,$$

invariant under the restriction of the geodesic flow ψ to $\mathrm{T}_{-1}M$, which defines an \mathbf{E} -bundle over Σ . Its fiber over the orbit $\Gamma\mathbf{v}$ of a fixed future-pointing unit-timelike vector \mathbf{v} is the union of geodesics in $M = \mathbf{E}/\Gamma$ parallel to $\Gamma\mathbf{v}$. In particular properness of the $\mathrm{L}(\Gamma)$ -action on \mathbf{H}^2 implies nonrecurrence of timelike geodesics, the last statement in Theorem 1.

More generally, any $\mathrm{L}(\Gamma)$ -invariant subset $\Omega \subset \mathbf{V}$ defines a subset $\mathrm{T}_\Omega(M) \subset \mathrm{T}M$ invariant under the geodesic flow. If Ω is an open set upon which $\mathrm{L}(\Gamma)$ acts properly, then the geodesic flow defines a proper \mathbb{R} -action on $\mathrm{T}_\Omega(M)$. In particular every geodesic whose velocity

lies in Ω is properly immersed and is neither positively nor negatively recurrent.

An important example is the following. The lines in S_0 form the *the ideal boundary*, (the circle-at-infinity) ∂H^2 , of H^2 . The *limit set* of $L(\Gamma)$ consists of endpoints of recurrent geodesic rays in Σ . Furthermore $\Lambda_{L(\Gamma)}$ is the unique closed $L(\Gamma)$ -invariant closed subset of ∂H^2 . In particular the set of fixed points of elements of $L(\Gamma)$ is dense in $\Lambda_{L(\Gamma)}$. Moreover $L(\Gamma)$ acts properly on the complement

$$\Omega := S_0 \setminus \Lambda_{L(\Gamma)}.$$

Applying the above discussion, no geodesic tangent to $T_\Omega(M)$ recurs. That is, a lightlike recurrent geodesic ray must be parallel to $\Lambda_{L(\Gamma)}$.

3. FROM GEODESICS IN Σ^2 TO GEODESICS IN M^3

While timelike directions correspond to points of Σ^2 , spacelike directions correspond to geodesics in H^2 . The recurrent geodesics in Σ intimately relate to the recurrent spacelike geodesics on M^3 .

Denote the set of oriented spacelike geodesics in E by \mathcal{S} . It identifies with the orbit space of the geodesic flow $\tilde{\psi}$ on $T_{+1}E \cong E \times S_{+1}$. The natural map $\mathcal{S} \xrightarrow{\sim} S_{+1}$ which associates to a spacelike vector its direction is equivariant respecting $\text{Isom}(E) \xrightarrow{L} \text{SO}(2, 1)$.

The identity component of $\text{SO}(2, 1)$ acts simply transitively on the unit tangent bundle UH^2 , and therefore we identify $\text{SO}(2, 1)^0$ with UH^2 by choosing a basepoint u_0 in UH^2 . Unit-spacelike vectors in S_{+1} correspond to oriented geodesics in H^2 . Explicitly, if $v \in S_{+1}$, then there is a one-parameter subgroup $a(t) \in \text{SO}(2, 1)$, having v as a fixed vector, and such that

$$\det(v, v^-, v^+) > 0,$$

where v^+ is an expanding eigenvector of $a(t)$ (for $t > 0$) and v^- is the contracting eigenvector. Choose a basepoint $v_0 \in S_{+1}$ corresponding to the orbit of u_0 under the geodesic flow on $U\Sigma$. Geodesics in H^2 relate to spacelike directions by an equivariant mapping

$$\begin{aligned} UH^2 &\longrightarrow S_{+1} \\ g(u_0) &\longmapsto g(v_0) \end{aligned}$$

The unit tangent bundle $U\Sigma$ of Σ identifies with the quotient

$$L(\Gamma) \backslash UH^2 \cong L(\Gamma) \backslash \text{SO}(2, 1)^0,$$

where the geodesic flow ψ corresponds the right-action of $a(-t)$ (see, for example, [15], §1.2).

Observe that a geodesic in Σ^2 is recurrent if and only if the endpoints of any of its lifts to $\tilde{\Sigma} \approx \mathbb{H}^2$ lie in the limit set $\Lambda_{\mathbf{L}(\Gamma)}$ of $\mathbf{L}(\Gamma)$. If the convex core of Σ^2 is compact, then the union $\mathbf{U}_{\text{rec}}\Sigma$ of recurrent ϕ -orbits is compact.

Lemma 2. *There exists an orbit-preserving map*

$$\mathbf{U}_{\text{rec}}\Sigma \xrightarrow{\hat{\mathbf{N}}} \mathbf{T}_{+1}(M)$$

mapping ϕ -orbits injectively to recurrent ψ -orbits.

Proof. The associated flat affine bundle \mathbb{E}_Γ over $\mathbf{U}\Sigma$ associated to the affine deformation Γ is defined as follows. The affine representation of Γ defines a diagonal action of Γ on $\widetilde{\mathbf{U}\Sigma} \times \mathbf{E}$. Its total space is the quotient of the product $\widetilde{\mathbf{U}\Sigma} \times \mathbf{E}$ by the diagonal action of $\pi_1(\mathbf{U}\Sigma)$:

$$\pi_1(\mathbf{U}\Sigma) \longrightarrow \pi_1(\Sigma) \longrightarrow \text{Isom}(\mathbf{E}).$$

Similarly the flat vector bundle \mathbf{V}_Γ over $\mathbf{U}\Sigma$ is the quotient of $\widetilde{\mathbf{U}\Sigma} \times \mathbf{V}$ by the diagonal action:

$$\pi_1(\mathbf{U}\Sigma) \longrightarrow \pi_1(\Sigma) \longrightarrow \text{Isom}(\mathbf{E}) \xrightarrow{\iota} \text{SO}(2, 1).$$

According to [15], the *neutral section* of \mathbf{V}_Γ is a $\text{SO}(2, 1)$ -invariant section which is parallel with respect to the geodesic flow on $\mathbf{U}\Sigma$. and arises from the graph of the $\text{SO}(2, 1)$ -equivariant map

$$\mathbf{U}\tilde{\Sigma} \cong \mathbf{U}\mathbb{H}^2 \longrightarrow \mathbf{V}$$

with image \mathbf{S}_{+1} , the space of unit-spacelike vectors in \mathbf{V} .

Here is the main construction of [15]. To every section σ of \mathbb{E}_Γ continuously differentiable along ϕ , associate the function

$$F_\sigma := \langle \nabla_\phi \sigma, \nu \rangle$$

on $\mathbf{U}\Sigma$. (Here the covariant derivative of a section of \mathbb{E}_Γ along a vector field ϕ in the base is a section of the associated vector bundle \mathbf{V}_Γ .) Different choices of section σ yield cohomologous functions F_σ . (Recall that two functions f_1, f_2 are *cohomologous*, written $f_1 \sim f_2$, if

$$f_1 - f_2 = \phi g$$

for a function g which is differentiable with respect to the vector field ϕ ([17], §2.2).

Restrict the affine bundle \mathbb{E}_Γ to $\mathbf{U}_{\text{rec}}\Sigma$. Lemma 8.4 of [15] guarantees the existence of a *neutralized section*, that is, a section N of $(\mathbb{E}_\Gamma)|_{\mathbf{U}_{\text{rec}}\Sigma}$ satisfying

$$\nabla_\phi N = f\nu,$$

for some function f .

Although the following lemma is well known, we could not find a proof in the literature. For completeness, we supply a proof in the appendix.

Lemma 3. *Let X be a compact space equipped with a flow ϕ . Let $f \in C(X)$, such that for all ϕ -invariant measures μ on X ,*

$$\int f \, d\mu > 0.$$

Then f is cohomologous to a positive function.

Since Γ acts properly, Proposition 8.1 of [15] implies that $\int F_\sigma d\mu \neq 0$ for all ϕ -invariant probability measures μ on $\mathbf{U}_{\text{rec}}\Sigma$. Since the set of invariant measures is connected, $\int F_\sigma d\mu$ is either positive for all ϕ -invariant probability measures μ on $\mathbf{U}_{\text{rec}}\Sigma$ or negative for all ϕ -invariant probability measures μ on $\mathbf{U}_{\text{rec}}\Sigma$. Conjugating by $-I$ if necessary we may assume that $\int F_\sigma d\mu > 0$. Lemma 3 implies $F_\sigma + \phi g > 0$ for some function g . Write

$$\widehat{N} = N + g\nu.$$

\widehat{N} remains neutralized, and $\nabla_\phi \widehat{N}$ vanishes nowhere.

Let $\widetilde{\mathbf{U}_{\text{rec}}\Sigma}$ be the preimage of $\mathbf{U}_{\text{rec}}\Sigma$ in \mathbf{UH}^2 . Then \widehat{N} determines a Γ -equivariant map

$$\widetilde{\mathbf{U}_{\text{rec}}\Sigma} \xrightarrow{\widehat{N}} \mathbf{E}.$$

Each $\tilde{\phi}$ -orbit injectively maps to a spacelike geodesic. The map

$$\begin{aligned} \mathbf{U}_{\text{rec}}\Sigma &\xrightarrow{\widehat{N}} (\mathbf{E} \times \mathbf{S}_{+1})/\Gamma \\ x &\longmapsto [(\widehat{N}(x), \nu(x))]. \end{aligned}$$

is the desired orbit equivalence $\mathbf{U}_{\text{rec}}\Sigma \longrightarrow \mathbf{T}_{+1}(M)$. \square

Lemma 4. *Any spacelike recurrent geodesic parallel to a geodesic γ in the image of \widehat{N} coincides with γ .*

Proof. Let $t \xrightarrow{g} \phi_t(v)$ be an orbit in $\mathbf{U}_{\text{rec}}\Sigma$. A geodesic ξ parallel to $\widehat{N}(g)$ determines a parallel section u of \mathbf{V} along g . Since g recurs, the resulting parallel section is a bounded invariant parallel section along the closure of g . By the Anosov property, such a section is along ν , and therefore, up to reparametrization, $\gamma = \widehat{N}(g)$. \square

Proposition 5. *\widehat{N} is injective and its image is the set of recurrent spacelike geodesics.*

Proof. An orbit of the geodesic flow ϕ recurs if and only if the corresponding Γ -orbit in the space \mathcal{S} of spacelike geodesics in \mathbb{E} recurs. Similarly a ϕ -orbit in $\mathbb{T}_{+1}(M)$ recurs if and only if the corresponding $\mathbb{L}(\Gamma)$ -orbit in \mathbb{S}_{+1} recurs. The map $\mathcal{S} \xrightarrow{\Upsilon} \mathbb{S}_{+1}$ recording the direction of a spacelike geodesic is \mathbb{L} -equivariant. If the Γ -orbit of $g \in \mathcal{S}$ corresponds to a recurrent spacelike geodesic in M , then the $\mathbb{L}(\Gamma)$ -orbit of $\Upsilon(g)$ corresponds to a recurrent ϕ -orbit in $\mathbb{U}\Sigma$.

$\widehat{\mathbb{N}}$ is injective along orbits of the geodesic flow. Thus it suffices to prove that the restriction of Υ to the subset of Γ -recurrent geodesics in \mathcal{S} is injective. Since the fibers of Υ are parallelism classes of spacelike geodesics, Lemma 4 implies injectivity of $\widehat{\mathbb{N}}$.

Finally let g be a ψ -recurrent point in $\mathbb{T}_{+1}(M)$, corresponding to a spacelike recurrent geodesic γ in M . It corresponds to a recurrent Γ -orbit Γg in \mathcal{S} . Then $\Upsilon(\Gamma g)$ is a recurrent $\mathbb{L}(\Gamma)$ -orbit in \mathbb{S}_{+1} , and corresponds to a recurrent ϕ -orbit in $\mathbb{U}\Sigma$. The image of this ϕ -orbit under $\widehat{\mathbb{N}}$ is a spacelike recurrent geodesic in $\mathbb{T}_{+1}(M)$ parallel to γ . Now apply Lemma 4 again to conclude that g lies in the image of $\widehat{\mathbb{N}}$. \square

The proof of Theorem 1 is complete.

4. APPENDIX: COHOMOLOGY AND POSITIVE FUNCTIONS

Let X be a smooth manifold equipped with a smooth flow ϕ . A function $g \in C(X)$ is *continuously differentiable along ϕ* if for each $x \in X$, the function

$$t \mapsto g(\phi_t(x))$$

is a continuously differentiable map $\mathbb{R} \rightarrow \mathbb{R}$. Denote the subspace of $C(X)$ consisting of functions continuously differentiable along ϕ by $C_\phi(X)$. For $g \in C_\phi(X)$, denote its directional derivative by:

$$\phi(g) := \left. \frac{d}{dt} \right|_{t=0} g \circ \phi_t.$$

The proof of Lemma 3 will be based on two lemmas.

Lemma 6. *Let $f \in C_\phi(X)$. For any $T > 0$, define*

$$f_T(x) := \frac{1}{T} \int_0^T f(\phi_s(x)) \, ds.$$

Then $f \sim f_T$.

Proof. We must show that there exists a function $g \in C_\phi(X)$ such that:

$$f_T - f = \phi g.$$

By the fundamental theorem of calculus,

$$f \circ \phi_t = f + \int_0^t (\phi f \circ \phi_s) \, ds.$$

Writing

$$g = \frac{1}{T} \int_0^T \int_0^t (f \circ \phi_s) \, ds \, dt,$$

then

$$\begin{aligned} f_T - f &= \frac{1}{T} \int_0^T (f \circ \phi_t - f) \, dt \\ &= \frac{1}{T} \int_0^T \int_0^t \phi(f \circ \phi_s) \, ds \, dt \\ &= \phi g. \end{aligned}$$

as desired. \square

Lemma 7. *Assume that for all ϕ -invariant measures μ ,*

$$\int f \, d\mu > 0.$$

Then $f_T > 0$ for some $T > 0$.

Proof. Otherwise sequences $\{T_m\}_{m \in \mathbb{N}}$ of positive real numbers and sequences $\{x_m\}_{m \in \mathbb{N}}$ of points in M exist such that

$$f_{T_m}(x_m) \leq 0.$$

Using the flow ϕ_t , push forward the (normalized) Lebesgue measure

$$\frac{1}{T_m} \mu_{[0, T_m]}$$

on the interval $[0, T_m]$ to X , to obtain a sequence of probability measures μ_n on X such that

$$\int f \, d\mu_n \leq 0.$$

As in [15], §7, a subsequence weakly converges to an ϕ -invariant measure μ for which

$$\int f \, d\mu \leq 0,$$

contradicting our hypotheses. \square

Proof of Lemma 3. By Lemma 6, $f \sim f_T$ for any $T > 0$, and Lemma 7 implies that $f_T > 0$ for some T . \square

REFERENCES

- [1] Abels, H., *Properly discontinuous groups of affine transformations, A survey*, Geom. Ded. **87** (2001) 309–333.
- [2] Abels, H., Margulis, G., and Soifer, G., *Properly discontinuous groups of affine transformations with orthogonal linear part.*, C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), no. 3, 253–258.
- [3] Abels, H., Margulis, G., and Soifer, G., *The linear part of an affine group acting properly discontinuously and leaving a quadratic form invariant*, Geom. Ded. (to appear).
- [4] Charette, V., *Proper Actions of Discrete Groups on $2+1$ Spacetime*, Doctoral dissertation, University of Maryland (2000).
- [5] Charette, V., *Affine deformations of ultraideal triangle groups*, Geom. Ded. **97** (2003), 17–31.
- [6] Charette, V., Drumm, T., and Goldman, W., *Affine deformations of a three-holed sphere*. Geom. Topol. **14** (2010), no. 3, 1355–1382,
- [7] Charette, V., and Goldman, W., *Affine Schottky groups and crooked tilings*. Crystallographic groups and their generalizations (Kortrijk, 1999), 69–97, Contemp. Math., 262, Amer. Math. Soc., Providence, RI, 2000.
- [8] Charette, V., Goldman, W., and Jones, C., *Recurrent Geodesics in Flat Lorentz 3-Manifolds*, Canad. Math. Bull. **47** (3), (2004), 332–342.
- [9] Drumm, T., *Linear holonomy of Margulis space-times*. J. Differential Geom. **38** (1993), no. 3, 676–690.
- [10] Drumm, T., *Fundamental polyhedra for Margulis space-times*. Doctoral dissertation, University of Maryland (1990); Topology **31** (1992), no. 4, 677–683.
- [11] Drumm, T. and Goldman, W., *Complete flat Lorentz 3-manifolds with free fundamental group*. Internat. J. Math. **1** (1990), no. 2, 149–161.
- [12] Eberlein, P., *Geodesic Flows on Negatively Curved Manifolds I*, Ann. Math. **95** (1972) No. 3, 492–510.
- [13] Fried, D., personal communication to W. Goldman (2002).
- [14] Fried, D. and Goldman, W., *Three-dimensional affine crystallographic groups*, Adv. Math. **47** (1983), 1–49.
- [15] Goldman, W., Labourie, F., and Margulis, G., *Proper affine actions and geodesic flows of hyperbolic surfaces*, Ann. Math. **170** No. 3, (2009), 1051–1083.
- [16] Goldman, W. and Margulis, G., *Flat Lorentz 3-manifolds and cocompact Fuchsian groups*, in “Crystallographic Groups and their Generalizations,” Contemp. Math. **262** (2000), 135–146, Amer. Math. Soc.
- [17] Katok, A. and Hasselblatt, B., “Introduction to the modern theory of dynamical systems. With a supplementary chapter by Katok and Leonardo Mendoza.” Encyclopedia of Mathematics and its Applications, **54**. Cambridge University Press, Cambridge (1995).
- [18] Labourie, F., *Fuchsian affine actions of surface groups*, J. Diff. Geo. **59** (1), (2001), 15 – 31.
- [19] Margulis, G., *Free properly discontinuous groups of affine transformations*, Dokl. Akad. Nauk SSSR **272** (1983), 937–940.
- [20] Margulis, G., *Complete affine locally flat manifolds with a free fundamental group*, J. Soviet Math. **134** (1987), 129–134.

- [21] Mess, G., *Lorentz spacetimes of constant curvature*, Geom. Ded. **126**, no. 1 (2007), 3-45, in “New techniques in Lorentz manifolds : Proceedings of the BIRS 2004 workshop,” (V. Charette and W. Goldman, eds.)
- [22] Milnor, J., *On fundamental groups of complete affinely flat manifolds*, Adv. Math. **25** (1977), 178–187.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK,
MD 20742 USA

E-mail address: wmg@math.umd.edu

UNIV. PARIS-SUD, LABORATOIRE DE MATHÉMATIQUES, ORSAY F-91405 CEDEX;
CNRS, ORSAY CEDEX, F-91405

E-mail address: francois.labourie@math.u-psud.fr